3. P. Eichoff, Principles of Control-System Identification [Russian translation], Mir, Moscow (1975).
4. D. F. Simbirskii and A. S. Gol'tsov, "Identification of nonsteady nonlinear thermal object using Kalman filter," Avtometriya, No. 1 (1975).
5. Experimental Thermal-Strength Methods for Gas-Turbine Motors [in Russian], Nos. 1 and 2, Khar'kov (1973).

DYNAMIC METHOD OF MEASURING HEAT FLUXES BY BATTERY HEAT FLOWMETERS USING A KALMAN FILTER

A. S. Gol'tsov, D. F. Simbirskii, and S. V. Kudryashov

UDC 536.629.7

A method is developed for the accelerated determination of stationary heat fluxes by battery heat flowmeters in the dynamic measurement mode.

Serially manufactured battery heat flowmeters [1], which are favorably distinguished by the simplicity of construction, the ease of fabrication, and the high response to the flux being measured, are used extensively in the practice of heat-flux measurement.

However, the thermal inertia of these heat flowmeters in certain cases of practical importance will either distort the results being obtained or increase substantially the total time for performing the experiment. From this viewpoint, methods of improving the dynamical characteristics of heat flowmeters, which are realized during subsequent processing of direct measurement, results by solving the inverse problem of heat conduction which occurs, are of indubitable interest. In particular, the problem of computing their values at the initial sections of the transient characteristic of the heat flowmeter can be posed in the measurement of stationary heat fluxes.

The solution of such a problem is described in [2] for the case of measuring the radiant heat flux by using a calorimetric heat flowmeter. The sensor of such a heat flowmeter is a single capacitance link whose dynamics is described by an ordinary differential equation. The magnitude of the heat flux was hence determined successfully by using the familiar Kalman-filter algorithm [3, 4]. However, direct application of a Kalman filter to the problem of measuring a stationary heat flux by battery heat flowmeters is impossible since their dynamics is described by the partial differential equation of heat conduction. At the same time, the Kalman filter is intended for an optimal estimation (in the sense of the root-mean-square deviation) of the state variables of dynamical systems with lumped parameters.

The algorithm of the Kalman filter is distinguished by its simplicity, is quite adaptable for realization on an electronic computer, takes account of the presence of random errors in the measurements, and processes information recurrently as it comes in. It can be applied to both linear and nonlinear dynamical systems. Moreover, utilization of the Kalman filter in the problem of determining the heat flux permits performance of a practical investigation of questions of the uniqueness and accuracy of the results obtained [2], which is especially important since the inverse heat-conduction problem to be solved is hence incorrectly posed.

In order to apply the Kalman filter to the problem of measuring heat fluxes by battery heat flowmeters, an approximate heat-meter model is proposed in this paper which is described by a system of ordinary differential equations and is obtained by the method of lines [5, 6].

In forming the mathematical model, the sensor of the heat flowmeter was considered as a finite rod, heat insulated along the side surface, and executed as a whole with a galvanic copper—Constantan differential thermal battery [1]. Some junctions of the thermal battery are brought out on the endface surface of the sensor which is fastened to the housing and whose temperature is measured by using a Chromel—Alumel thermocouple. Other junctions are disposed on a plane removed a distance $B = 0.2 \cdot 10^{-3}$ m from the detecting surface of the sensor.

Khar'kov Aviation Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 33, No. 6, pp. 1070-1077. Original article submitted April 5, 1977.

Because of the use of effective cooling, the sensor temperature during the measurements ordinarily varies within a narrow band [1]. It can hence be assumed that the thermophysical characteristics of the sensor material are independent of the temperature and the characteristics of the thermocouples being used are linear.

Under the assumptions made above, the heat-transfer process in the sensor bulk is described by the linear equation of heat conduction

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial x^2}, \quad 0 \leqslant \tau \leqslant \tau_N, \quad 0 \leqslant x \leqslant l \tag{1}$$

with the boundary conditions

$$T(0, x) = 0, \quad 0 \leqslant x \leqslant l, \tag{2}$$

$$\lambda \frac{\partial T(\tau, 0)}{\partial x} = -q, \quad 0 \leqslant \tau \leqslant \tau_N, \tag{3}$$

$$\alpha_1 T(\tau, l) = V(\tau) + \varepsilon(\tau), \tag{4}$$

where $V(\tau)$ is the output signal of the Chromel—Alumel thermocouple.

The thermal-battery output is related to the temperature of the sites of its junctions as follows:

$$W(\tau) = \alpha_2 (T(\tau, B) - T(\tau, l)). \tag{5}$$

By using the method of lines [5, 6], Eqs. (1)-(5) can be approximated by the system of ordinary differential equations

$$\frac{dt_1}{d\tau} = \dot{t}_1 = \frac{2a}{h^2}(t_2 - t_1) + \frac{2}{c\rho h} q,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dot{t}_i = \frac{a}{h^2}(t_{i-1} + t_{i+1} - 2t_i), \tag{6}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dot{t}_n = \frac{a}{h^2}(t_{n-1} + t_j - 2t_n) - \frac{a}{\alpha_2 h^2} W(\tau)$$

with the initial conditions

$$t_i(0) = 0, \quad i = 1, 2, \dots, n. \tag{7}$$

In deriving (6) it was taken into account that (4) can be written in the form

$$V(\tau) = \alpha_1 t_j - \frac{\alpha_1}{\alpha_2} W(\tau) + \varepsilon(\tau), \tag{8}$$

where $t_j = T(\tau, B)$.

Let us linearize (8) at the time τ_k with respect to an estimate of the heat flux q_{k-1} obtained at the preceding estimation step:

$$Y(\tau_k) = V(\tau_k) + \frac{\alpha_1}{\alpha_2} W(\tau_k) \approx \alpha_1 t_j(\tau_k, \hat{q}_{k-1}) + \alpha_1 U_j(\tau_k)(q - \hat{q}_{k-1}). \tag{9}$$

The response of the temperature to the heat flux is determined from the solution of the response differential equations

$$\dot{U}_1 = \frac{2a}{h^2}(U_2 - U_1) + \frac{2}{c\rho h},$$

$$\dots\dots\dots\dots\dots\dots$$

$$\dot{U}_i = \frac{a}{h^2}(U_{i-1} - 2U_i + U_{i+1}), \tag{10}$$

. . . .

$$\dot{U}_n = \frac{a}{h^2}\,(U_{n-1}+U_j-2\,U_n)$$

with the initial conditions

$$U_i(0)=0,\ i=1,2,\ldots,\ n. \tag{11}$$

It follows from the condition of stationarity of the heat flux that

$$q(\tau_{k+1})=q(\tau_k). \tag{12}$$

Therefore, (6), (7), (9), (10), (11), and (12) will describe the **mathematical model of the heat flowmeter** under consideration.

We determine the magnitude of the unknown heat flux by minimizing the following quadratic quality function [4]:

$$\Phi(q)=\frac{\sigma^2}{N}\sum_{k=1}^{N}[Y(\tau_k)-\alpha_1 t_j(\tau_k,q)]^2.$$

Under the assumptions made above, the value $\hat{q}$ which yields the minimum of the quality function $\Phi(q)$ and later called the optimal estimate of the heat flux q, can be obtained by means of the algorithm

$$\hat{q}(\tau_{k+1})=\hat{q}(\tau_k)+\frac{\alpha_1}{\sigma^2}P_{k+1}U_j(\tau_{k+1})\Big[V(\tau_{k+1})+\frac{\alpha_1}{\alpha_2}W(\tau_{k+1})-\alpha_1 t_j(\tau_{k+1},\hat{q}(\tau_k))\Big], \tag{13}$$

$$P_{k+1}=\frac{\sigma^2 P_k}{\sigma^2+\alpha_1 U_j(\tau_k)P_k}, \tag{14}$$

whose detailed derivation is presented in [4] in the more general case.

The $U_j(\tau)$ in (13) and (14) is determined as a result of solving the system (10), (11) while $t_j(\tau, \hat{q}(\tau_k))$ is evaluated at each step of the estimation by means of (6) and (7) for $q=\hat{q}(\tau_k)$.

The estimation algorithm obtained for the heat flux is a modification of the discrete Kalman-filter algorithm [3, 4] and differs from this latter by the fact that instead of estimating the state vector of dimensionality (n + 1) and solving the Riccati matrix equation, scalar quantities are used in (13) and (14) to calculate the variance.

An initial estimate of the heat flux $\hat{q}(0)$ by considering it as a random variable, as well as the variance P_0 of this estimate, must be given in order to evaluate the heat flux by means of (13) and (14).

In a practical realization of the algorithm (13), (14), (6), (7), (10), and (11) the number of nodes of the difference mesh n in (6) and (10) must be selected after analyzing the influence of errors of the method of lines on the accuracy in determining the heat flux. The analysis can be performed by the method elucidated in [8]. The error in finding the optimal estimates of the heat flux related to the errors in the method of lines can be calculated by means of the equation

$$\delta q=-\sum_{k=1}^{N}\frac{\sum\limits_{i=1}^{n}U_i(\tau_k)\Delta_i(\tau_k,0)}{\sum\limits_{i=1}^{n}U_i^2(\tau_k)}, \tag{15}$$

where $U_i(\tau)$ are the sensitivity functions determined by means of (10) and (11).

The errors of the method of lines $\Delta_i(\tau, x_i)$ can be found as follows. The dependence of the temperature on the space coordinate in the section $0 \le x_i \le h$ is approximated in the method of lines by a truncated Taylor series [6] which has the following form after replacement of the partial derivatives by their difference analogs:

$$T(\tau, x_1) = t_1 - \frac{1}{\lambda} q x_1 + \left(t_2 - t_1 + \frac{h}{\lambda} q\right) \frac{x_1^2}{h^2} + \Delta_1(\tau, x_1),$$
$$T(\tau, x_i) = t_2 + (t_{i+1} - t_{i-1}) \frac{x_i}{2h} + (t_{i-1} - 2t_i + t_{i+1}) \frac{x_i^2}{h^2} + \Delta_i(\tau, x_i). \tag{16}$$

After substituting the right sides of (16) into (1)-(5) and performing appropriate manipulations with (6) taken into account for the errors in the method of lines, the following system of partial differential equations can be obtained:

$$\frac{\partial \Delta_1}{\partial \tau} = a \frac{\partial^2 \Delta_1}{\partial x_1^2} + \frac{5a}{6h^2}(3t_1 - 4t_2 + t_3) - \frac{5}{3c\rho h} q,$$
$$\frac{\partial \Delta_2}{\partial \tau} = a \frac{\partial^2 \Delta_2}{\partial x_2^2} + \frac{a}{12h^2}(2t_1 - 11t_2 + 14t_3 - 5t_4) + \frac{1}{6c\rho h} q,$$
$$\frac{\partial \Delta_i}{\partial \tau} = a \frac{\partial^2 \Delta_i}{\partial x_i^2} + \frac{a}{12h^2}(t_{i-2} + 2t_{i-1} - 12t_i + 14t_{i+1} - 5t_{i+2}), \tag{17}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial \Delta_{n-1}}{\partial \tau} = a \frac{\partial^2 \Delta_{n-1}}{\partial x_{n-1}^2} + \frac{a}{12h^2}(t_{n-3} + 2t_{n-2} - 12t_{n-1} + 14t_n - 55t_j + \alpha_2^{-1} W(\tau)),$$
$$\frac{\partial \Delta_n}{\partial \tau} = a \frac{\partial^2 \Delta_n}{\partial x_n^2} + \frac{a}{12h^2}(t_{n-2} + 2t_{n-1} - 7t_n + 4t_j - 4\alpha_2^{-1} W) - \frac{5}{12}\alpha_2^{-1}\dot{W}$$

with the boundary conditions

$$\Delta_i(\tau, h) = 0, \; i = 1, 2, \ldots, n, \quad \frac{\partial \Delta_1(\tau, 0)}{\partial x_1} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_1(\tau, h) = \Delta_2(\tau, 0), \quad \frac{\partial \Delta_1(\tau, h)}{\partial x_1} = \frac{\partial \Delta_2(\tau, 0)}{\partial x_2} + \frac{1}{h}(3t_1 - 4t_2 + t_3) - \frac{1}{\lambda} q,$$
$$\Delta_i(\tau, h) = \Delta_{i+1}(\tau, 0), \quad \frac{\partial \Delta_i(\tau, h)}{\partial x_i} = \frac{\partial \Delta_{i+1}(\tau, 0)}{\partial x_{i+1}} + \frac{1}{2h}(t_{i-1} - t_i - t_{i+1} + t_{i+2}), \tag{18}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{n-1}(\tau, h) = \Delta_n(\tau, 0), \quad \frac{\partial \Delta_{n-1}(\tau, h)}{\partial x_{n-1}} = \frac{\partial \Delta_n(\tau, 0)}{\partial x_n} + \frac{1}{2h}(t_{n-2} - t_{n-1} - t_n + \alpha_2^{-1} W),$$
$$\Delta_n(\tau, h) = 0.$$

Using the differential-difference approximation (17) and (18) were solved by the method of lines jointly with (1)-(4) by the Runge–Kutta method for different n and the values q = 1 and W = 0. The values of the relative error of the method of lines obtained here were used to calculate their corresponding relative errors of the optimal heat-flux estimates by means of (15). Results of the calculations are presented in Fig. 1 as a function of the dimensionless time Fo $= a\tau/l^2$ for n = 3 and n = 5. It follows from the figure that the magnitude of the relative error in determining the heat flux diminishes with time and with an increase in the number of approximation nodes. For Fo ≥ 0.2 it hence becomes less than 5% in the case n = 5.

The algorithm obtained above was used for a practical determination of the stationary heat flux by using a battery heat flowmeter characterized by the following parameters [1]:

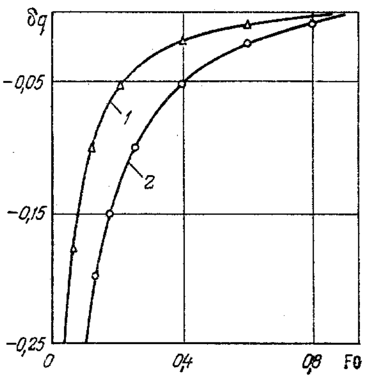

Fig. 1

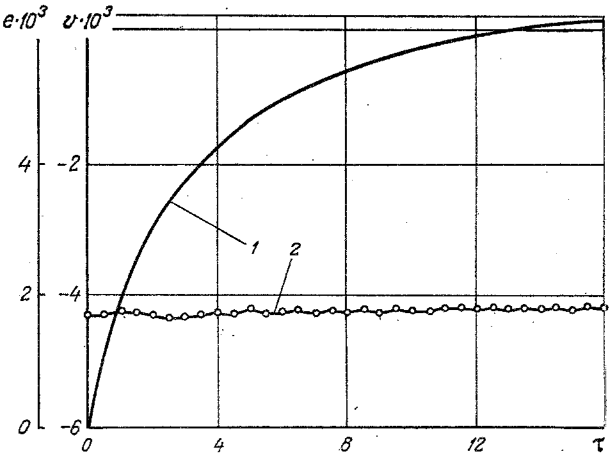

Fig. 2

Fig. 1. Relative error of the optimal heat-flux estimates as a function of the number of approximation nodes (δq, Fo): 1) n = 5, 2) n = 3.

Fig. 2. Dependence of the heat-flowmeter output signals on the measurement time: 1) output signal of the thermal battery (e, V; τ, sec); 2) output signal of the Chromel—Alumel thermocouple (v, V; τ, sec).

$$l = 1.2 \cdot 10^{-3}\,\mathrm{m};\; a = 0.21 \cdot 10^{-6}\,\mathrm{m^2 \cdot sec^{-1}},\; \alpha_1 = 3.2 \cdot 10^{-5}\,\mathrm{V \cdot {}^\circ K};$$
$$\alpha_2 = 0.0515\,\mathrm{V \cdot {}^\circ K}.$$

The heat-flowmeter output signals, inscribed on a loop oscilloscope, are presented in Fig. 2.

It has been established as a result of the measurements that the time constant of the heat flowmeter under consideration is $\tau_{0.63} = 3.5$ sec, while the time of its emergence in the steady-state mode is greater than 40 sec.

Since the Fo = 0.02 in the case under consideration corresponds to the real time $\tau = 1$ sec, then it follows from Fig. 1 that to assure an error in determining the heat flux (associated with the error in the method of lines) of less than 5%, the number of approximation nodes in (6) and (10) can be taken equal to 5. Hence j = 2 in (6), (1), (13), and (14).

The heat flux was evaluated by means of (13), (14), (6), (7), (10), and (11) for the given initial estimates: $\hat{q}(0) = 1000$; 1200 and 1500 W/m² with the variance $P_a = 10^{16}$ W²/m⁻⁴. The variance of the measurement of the quantity $\sigma^2 = 9 \cdot 10^{-10}$ V². The calculations were performed recurrently as the measurements taken discretely off from the oscillogram with the time spacing $\Delta\tau = 0.5$ sec came in.

The results of calculating the heat-flux estimates are presented in Fig. 3. As follows from the figure, the optimal estimation procedure is stable and convergent. The final value of the magnitude of the heat flux was 1130 W/m². This quantity was indeed taken as the true value of the desired heat flux.

In addition, the confidence range of the estimates obtained, evaluated by means of the equation [4]

$$\Delta q_k = \pm 3\sqrt{P_k} \tag{19}$$

with probability 0.99, is presented in Fig. 3.

The heat flux in the steady-state mode was measured by using the calibration characteristics of the heat flowmeter in order to estimate the accuracy of the results obtained. The magnitude of the heat flux hence obtained, which equals 1250 W/m², is presented in Fig. 3, from which it is seen that it lies within the confidence range calculated by means of (19).

It is seen from Fig. 3 that the value of the heat flux which differs by not more than 3% from the final estimate (equal to 1130 W/m²) has been obtained during processing the results of measurements performed during the first 3 sec after the beginning of the experiment.

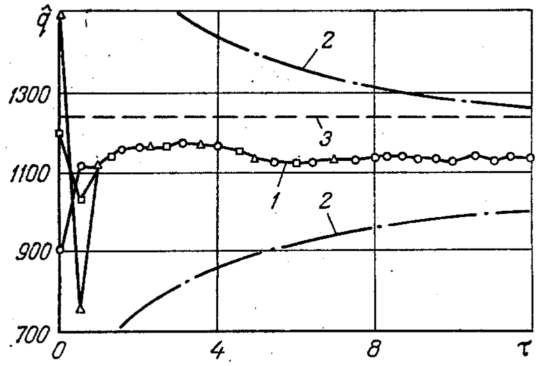

Fig. 3. Results of an experimental—computational determination of the heat flux ($\hat{q}$, W/m^2; τ, sec): 1) recurrent estimate of the heat flux by means of (14); 2) magnitude of the heat flux found in the steady-state mode by using the calibration characteristics of the heat flowmeter; 3) confidence range of the heat-flux estimates.

Therefore, by using the mathematical apparatus of the theory of sensitivity functions, a method has been developed for recurrent determination of the heat flux in the dynamical mode of measurements of a battery heat flowmeter, which is a modification of the Kalman-filter algorithm of minimal dimensionality.

A method has been developed for the practical determination of errors of the method of lines, which is used in composing the mathematical model of the heat flowmeter, and an analysis is also performed on the influence of these errors on the accuracy of obtaining finite results.

The use of the developed dynamic method in the problem of a practical measurement of the heat flux permitted the diminution of the actual inertia of the heat flowmeter used by more than tenfold. The results hence obtained agree well with the magnitude of the heat flux obtained by direct measurement in the steady-state mode.

NOTATION

T, t_η, temperature; x space coordinate; τ, time; l, sensor length; q, specific heat flux; λ, coefficient of thermal conductivity; c, specific heat; ρ, density; a, coefficient of thermal diffusivity; α_1, thermoelectric coefficient of the galvanic copper—Constantan thermal battery; α_2, thermoelectric coefficient of the Chromel—Alumel thermocouple; ε, error in temperature measurement; P, variance of the heat-flux estimate; $\Delta(\tau, x)$, error in determining the temperature field by the method of lines; $U_i(\tau)$, temperature response to the heat flux being measured; $h = l/n$, space quantization step in the method of lines.

LITERATURE CITED

1. O. A. Gerashchenko, Principles of Heat Measurement [in Russian], Naukova Dumka, Kiev (1971).
2. A. V. Oleinik, Experimental Methods of the Heat Strength of Gas Turbine Engines [in Russian], Vol. 2, Khar'kov Aviation Inst., Khar'kov (1975), p. 189.
3. R. Kalman and R. Bucy, Trans. ASME, Ser. D, Eng. Mech. [Russian translation], 83, No. 1, 123 (1961).
4. D. F. Simbirskii, Temperature Diagnostics of Engines [in Russian], Tekhnika, Kiev (1976).
5. L. A. Kozdoba, Methods of Solving Nonlinear Heat-Conduction Problems [in Russian], Nauka, Moscow (1975).
6. B. A. Volynskii and V. E. Bukhman, Models for Boundary-Value-Problem Solution [in Russian], Izd. Fiz.-Mat. Lit., Moscow (1960).
7. V. I. Gorodetskii, F. M. Zakharin, E. N. Rozenvasser, and R. M. Yusupov, Sensitivity Methods in Automatic-Control Theory [in Russian], Énergiya, Leningrad (1971).
8. A. S. Gol'tsov and D. F. Simbirskii, Experimental Methods of the Heat Strength of Gas Turbine Engines [in Russian], Vol. 2, Khar'kov Aviation Institute, Khar'kov (1975), p. 16.